Phase models with explicit time delays

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Studying weakly connected oscillators leads to phase models. It has been proven recently that weakly connected oscillators with delayed interactions *do not* lead to phase models with time delays even when the delay is of the same order of magnitude as the period of oscillation. This has resulted in a fading interest in such models. We prove here that if the interaction delay between weakly connected oscillators is much longer than the period of oscillation, then the corresponding phase model *does have* an explicit time delay. [S1063-651X(98)10307-0]

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I. INTRODUCTION

Many phenomena in biology, chemistry, and engineering can be described by a network of oscillators. The most spectacular example is a synchronous rhythmic flashing of fireflies [1]. Among many other examples (see [2] and references therein) we mention synchronization of pacemaker cells of the heart [3], chemical waves [4], and rhythmic activity in the brain [5]. The latter has a prominent feature: the interaction delay between the oscillators can be as large as the oscillation period. Thus, the question is, *can such a delay endow the oscillatory network with new dynamical features?* In this paper we show that if the oscillators are weakly connected and the delay has the same magnitude as the period of oscillation, then it does not play any role and can be neglected. In contrast, the delay starts to play a significant role when it is comparable with $1/\varepsilon$ periods, where $\varepsilon \ll 1$ is the strength of connections.

A. Phase models

There is an intimate relationship between weakly connected oscillators and phase models. Namely, for any weakly connected network of oscillators of the form

$$\frac{dx_i}{dt} = f_i(x_i) + \varepsilon g_i(x_1, \ldots, x_n), \quad x_i \in \mathbb{R}^m, \quad \varepsilon \ll 1, \quad (1)$$

there is a continuous noninvertible local mapping $p: \mathbb{R}^{nm} \times \mathbb{R} \to \mathbb{T}^n$, where $\mathbb{T}^n$ is the n-torus, that projects solutions of Eq. (1) to those of the phase model of the form

$$\frac{d\theta_i}{dt} = \Omega_i + \varepsilon h_i(\theta_1, \ldots, \theta_n, \varepsilon),$$

where $\Omega_i \in \mathbb{R}$ is the frequency of the ith oscillator, $\theta_i \in \mathbb{S}^1$ is its phase, and $\mathbb{S}^1$ is the unit circle [6]. Since many systems of the form (1) can be transformed into the phase model above by a continuous change of variables, the phase model is referred to as being *canonical*. Numerous other examples of canonical models can be found in [6].

When the oscillators have identical frequencies

$$\Omega_1 = \cdots = \Omega_n = \Omega,$$

then the projection p could be chosen so that the phase model has the form

$$\frac{d\theta_i}{dt} = \Omega + \varepsilon h_i(\theta - \theta_i) + O(\varepsilon^2),$$

where $\theta = (\theta_1, \ldots, \theta_n)^\top \in \mathbb{T}^n$ and $\theta - \theta_i = (\theta_1 - \theta_i, \ldots, \theta_n - \theta_i)^\top$; see, e.g., [6–8,4].

To study dynamics of the phase model it is convenient to introduce the phase deviation variables $\varphi_i \in \mathbb{S}^1$, so that

$$\theta_i(t) = \Omega t + \varphi_i.$$

Then the phase model can be written in the form

$$\frac{d\varphi_i}{dt} = \varepsilon h_i(\varphi - \varphi_i) + O(\varepsilon^2).$$

The key observation here is that the phase deviations φ_i are slow variables: We introduce the slow time $\tau = \varepsilon t$ and rewrite the system above in the form

$$\frac{d\varphi_i}{d\tau} = h_i(\varphi - \varphi_i) + O(\varepsilon). \quad (2)$$

Much research keeps only an initial portion of the Fourier series of the functions h_i, which leads to the well-known Kuramoto phase model,

$$\frac{d\varphi_i}{d\tau} = \omega_i + \sum_{j=1}^{n} s_{ij} \sin(\varphi_j + \psi_{ij} - \varphi_i).$$

Here each parameter has a well-established meaning: ω_i is the frequency deviation of the ith oscillator, s_{ij} encodes the strength of connection from the jth to the ith oscillator, and each $\psi_{ij} \in \mathbb{S}^1$ is the natural phase difference [6].

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B. Phase models with delays

There are a number of papers [9–12] where Kuramoto's model is considered with an explicit time delay

$$\frac{d\varphi_i(\tau)}{d\tau}=\omega_i+\sum_{j=1}^{n} s_{ij}\ \sin[\varphi_j(\tau-\eta)+\psi_{ij}-\varphi_i(\tau)],$$

which takes into account the finite speed of interactions between the oscillators. This is definitely the case in the neuroscience applications, since the transmission via nonmyelin axons is very slow [13], and the delay may be significant (in comparison with the interspike period).

It is reasonable to assume that phase models with delays are canonical for weakly connected oscillators with explicit transmission delays,

$$\frac{dx_i}{dt}=f_i(x_i(t))+\varepsilon g_i(x_1(t-\eta),\ldots,x_n(t-\eta)). \tag{3}$$

The phase model for such a network was derived in [14,6] (see also Corollary 2 below), and it turned out to be without any explicit delay, but only with an additional natural phase shift ψ; i.e.,

$$\frac{d\varphi_i}{d\tau}=h_i(\varphi-\psi-\varphi_i)+O(\varepsilon),$$

where $\psi=\eta\Omega \bmod 2\pi$, and the functions h_i are the same as in Eq. (2).

Without elaborating how the phase shift appears, let us discuss how the explicit time delay disappears. For this, notice that the phase deviation variables $\varphi(\tau)$ depend on the slow time $\tau=\varepsilon t$. Therefore,

$$\varphi(\varepsilon(t-\eta))=\varphi(\tau-\varepsilon\eta)=\varphi(\tau)+O(\varepsilon\eta),$$

and the explicit time delay η does disappear when $\eta=\text{const}$, but $\varepsilon\ll1$; that is, when the transmission delay is of the same order of magnitude as the period of oscillation. This is the case considered in [14,6]. Obviously, the fact that finite transmission speed creates only a simple phase shift in the coupling functions h_i could undermine the significance of studying phase models with delays.

In this paper we prove that if the transmission delay is long enough, i.e., if it is comparable with $1/\varepsilon$ cycles, then the phase model does acquire an explicit time delay. This should revive the interest in such phase models.

II. MAIN RESULT

Consider a network of weakly connected oscillators of the form (3) having nearly identical frequencies. Without loss of generality we may assume that the frequency is 1. The following theorem is a generalization of the Malkin theorem for weakly connected oscillators; see [15–17] or the book by Hoppensteadt and Izhikevich [6], who proved the theorem following [7,8].

Theorem (phase model for weakly connected oscillators with transmission delays). Consider a weakly connected system of the form (3), and suppose that each equation in the uncoupled system

$$\frac{dx_i}{dt}=f_i(x_i),\quad x_i\in\mathbb{R}^m, \tag{4}$$

has an exponentially orbitally stable 2π-periodic solution $\gamma_i\subset\mathbb{R}^m$. In this case the system (3) has a normally hyperbolic invariant manifold $M=\gamma_1\times\cdots\times\gamma_n$ for $\varepsilon=0$. Suppose that the manifold persists under the ε-perturbations for $\varepsilon>0$.

Let $\tau=\varepsilon t$ be slow time and let $\varphi_i(\tau)\in\mathbb{S}^1$ be the phase deviation from the natural oscillation $\gamma_i(t), t\geqslant0$. Then, there is an $\varepsilon_0>0$ such that for all $0<\varepsilon\leqslant\varepsilon_0$ the vector of phase deviations $\varphi=(\varphi_1,\ldots,\varphi_n)^\top\in\mathbb{T}^n$ is a solution to

$$\frac{d\varphi_i(\tau)}{d\tau}=h_i[\varphi(\tau-\zeta)-\psi-\varphi_i(\tau)]+\mathcal{O}(\varepsilon), \tag{5}$$

where

$$\zeta=\varepsilon\eta\quad\text{and}\quad\psi=\eta\mod 2\pi,$$

and h_i are some 2π-periodic functions.

Proof. Hoppensteadt and Izhikevich (Lemma 4.5 in [6]) proved that a direct product of exponentially asymptotically stable limit cycle attractors, $M=\gamma_1\times\cdots\times\gamma_n$, is a normally hyperbolic compact invariant manifold for the uncoupled system (4) (this may not be true for direct products of other normally hyperbolic invariant manifolds). Since that invariant manifold persists for $\varepsilon>0$, there is $\varepsilon_0>0$ such that for all $\varepsilon\leqslant\varepsilon_0$ system (3) has a normally hyperbolic invariant manifold in an ε-neighborhood of M. Let us denote

$$x_i(t)=\gamma_i[t+\phi_i(\tau)]+\varepsilon P_i(t,\phi,\varepsilon),$$

where smooth vector functions εP_i account for the ε-perturbation of the invariant manifold M. Our goal below is to find the equations for $\varphi_i(\tau)$. For the sake of convenience we denote $\varphi_i(\tau)$ simply by φ_i.

We differentiate the equation above with respect to t to obtain

$$\begin{aligned}\frac{dx_i}{dt}&=\gamma_i'(t+\varphi_i)\left(1+\varepsilon\frac{d\varphi_i}{d\tau}\right)+\varepsilon\frac{\partial P_i(t,\varphi,\varepsilon)}{\partial t}\\&=f_i[\gamma_i(t+\varphi_i)]+\varepsilon g_i\{\gamma[t-\eta+\varphi(\tau-\zeta)]\}\\&\quad+\varepsilon Df_i[\gamma_i(t+\varphi_i)]\ P_i(t,\varphi,\varepsilon)\end{aligned}$$

plus terms of order $O(\varepsilon^2)$. Since

$$\gamma_i'(t+\varphi_i)=f_i[\gamma_i(t+\varphi_i)],$$

we obtain

$$\begin{aligned}f_i[\gamma_i(t+\varphi_i)]\frac{d\varphi_i}{d\tau}+\frac{\partial P_i(t,\varphi,0)}{\partial t}&\\=g_i\{\gamma[t-\eta+\varphi(\tau-\zeta)]\}&\\+Df_i[\gamma_i(t+\varphi_i)]\ P_i(t,\varphi,0).&\end{aligned}$$

It is convenient to rewrite the equation above in the form

$$\frac{dy_i(t,\varphi)}{dt}=A_i(t,\varphi_i)\ y_i(t,\varphi)+b_i(t,\varphi), \tag{6}$$

where $y_i(t,\varphi)=P_i(t,\varphi,0)$ is an unknown vector variable. The matrix

$$A_i(t,\varphi_i)=Df_i[\gamma_i(t+\varphi_i)]$$

and the vector

$$b_i(t,\varphi)=g_i\{\gamma[t-\eta+\varphi(\tau-\zeta)]\}-f_i[\gamma_i(t+\varphi_i)]\frac{d\varphi_i}{d\tau}$$

are 2π-periodic in t. Both vectors φ and $\varphi(\tau-\zeta)$ are treated as parameters here.

To study existence and uniqueness of solutions to Eq. (6) one must consider the *adjoint* linear homogeneous system

$$\frac{dq_i(t,\varphi_i)}{dt}=-A_i(t,\varphi_i)^\top q_i(t,\varphi_i) \tag{7}$$

with a normalization condition, which we take in the form

$$\frac{1}{2\pi}\int_0^{2\pi} q_i(t,\varphi_i)^\top f_i[\gamma_i(t+\varphi_i)]\,dt=1. \tag{8}$$

Each limit cycle γ_i is exponentially orbitally stable. Hence, a homogeneous ($b\equiv 0$) linear system of the form (6) and the adjoint system (7) have 1 as a simple Floquet multiplier, and the other multipliers are not on the unit circle. This implies, in particular, that the adjoint system (7) has a unique nontrivial periodic solution, say $q_i(t,\varphi_i)$, which can easily be found using standard numerical methods. Now we use the Fredholm alternative to conclude that the linear system (6) has a unique periodic solution $y_i(t,\varphi)$ if and only if the *orthogonality condition*

$$\langle q,b\rangle=\frac{1}{2\pi}\int_0^{2\pi} q_i(t,\varphi_i)^\top b_i(t,\varphi)\,dt=0$$

holds. Due to the normalization condition (8) and expression for b_i, this is equivalent to

$$\frac{d\varphi_i}{d\tau}=\frac{1}{2\pi}\int_0^{2\pi} q_i(t,\varphi_i)^\top g_i\{\gamma[t-\eta+\varphi(\tau-\zeta)]\}\,dt.$$

Due to the special form of the matrix $A_i(t,\varphi_i)$, it suffices to find a solution $q_i(t,\varphi_i)$ to the adjoint system (7) for $\varphi_i=0$, and any other solution $q_i(t,\varphi_i)$ has the form $q_i(t,\varphi_i)=q_i(t+\varphi_i,0)$, which we denote simply by $q_i(t+\varphi_i)$. Now we rewrite the equation above in the form

$$\frac{d\varphi_i}{d\tau}=\frac{1}{2\pi}\int_0^{2\pi} q_i(t+\varphi_i)^\top G_i\{\gamma[t-\eta+\varphi(\tau-\zeta)]\}\,dt$$

or in the form

$$\frac{d\varphi_i}{d\tau}=\frac{1}{2\pi}\int_0^{2\pi} q_i(s)^\top G_i\{\gamma[s-\eta+\varphi(\tau-\zeta)-\varphi_i]\}\,ds,$$

which implies Eq. (5), where

$$h_i(\phi)=\frac{1}{2\pi}\int_0^{2\pi} q_i(s)^\top G_i[\gamma(s+\phi)]\,ds.$$

Remark. The assumption that the invariant manifold M persists under the time delayed perturbations is not necessary when $\eta=0$ (Fenichel [18]) or even when $\eta=$ const (Hirsch *et al.* [19]). *Whether or not it is necessary when $\eta=O(1/\varepsilon)$ is still an open question.* Indeed, differential delay equations have very nice flows in the Banach manifold of maps of the compact interval $[0,\eta]$ into the n-torus M [Morris Hirsch (personal communication)], but the interval $[0,\eta]$ becomes unbounded as $\varepsilon\to 0$. So far we can neither prove the persistence of M nor present a counterexample.

The following corollary constitutes the major result of this paper.

Corollary 1 (long transmission delay). If the transmission delay η in the weakly connected system (3) is long enough, that is, if it has order of magnitude of $1/\varepsilon$ periods, then the delay term in the phase equation (5) persists.

Proof. The proof follows from the fact that $\zeta=\varepsilon\eta=O(1)$ in the phase model (5).

The following corollary is due to [14,6].

Corollary 2 (short transmission delay). If the transmission delay in the weakly connected system (3) is comparable with the period of an oscillation, that is, if $\eta=O(1)$, then the phase equation (5) can be written in the form

$$\frac{d\varphi_i}{d\tau}=h_i(\varphi-\psi-\varphi_i)+O(\varepsilon).$$

Proof. From $\zeta=\varepsilon\eta=O(\varepsilon)$, it follows that $\varphi(\tau-\zeta)=\varphi(\tau)+O(\varepsilon)$, and, hence, the delay term affects only the small remainder $O(\varepsilon)$ in the phase model above.

One should be warned that the remainder $O(\varepsilon)$ in the phase models could be discarded only when the truncated system

$$\frac{d\varphi_i}{d\tau}=h_i(\varphi-\psi-\varphi_i)$$

frequency-locks; that is, when $\varphi(\tau)$ approaches a hyperbolic limit cycle attractor (for the definitions of frequency locking, phase locking, synchronization, etc., see Chap. 9 in [6]). If it does not, the remainder may affect the dynamics significantly, and hence cannot be neglected.

III. DISCUSSION

In this paper we study how a finite transmission speed affects weakly connected oscillators. We confirmed the result [14] that this always induces a natural phase shift into the phase model provided that the delay is not a multiple of a period.

When the delay in the weakly connected system is comparable with one or a few periods, no delayed term appears in the phase model. In contrast, when the transmission delay is comparable with many ($1/\varepsilon$) periods, the phase model does acquire an explicit time delay term. This may lead to rich and complicated dynamics even when there are only two weakly connected oscillators [9].

It should be stressed that the absolute value of the delay is not important. Only its relative size to the period matters. Thus, the axon transmission delay of, say, 100 ms induces only a natural phase shift between periodically spiking neurons with the frequency 10 Hz, but the same transmission delay may be important and could not be removed from the phase equations if the neurons fire with the frequency 100 Hz, and the strength of connections is $\varepsilon\approx 0.1$.

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